The Firm-Value Risk Model

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Abstract. The Firm-Value Risk Model combines the technology of actuarial optimal dividends models with insights regarding financial frictions from financial economics, especially as they apply to risk transfer in (re)insurance firms. This paper illustrates, by numerical solution of a set of case studies, how certain stylized facts about (re)insurer value emerge naturally from the Firm-Value Risk Model, specifically: (1) the concave relationship between firm value and financial slack (2) the convex relationship between firm value and risk, (3) the substitution of external capital for reinsurance, and (4) the irrelevance of risk management when external capital is freely available, even in the presence of customer risk aversion.

Keywords. Financial friction, Modigliani-Miller, Froot-Stein, de Finetti, optimal dividends, risk management, reinsurance, shareholder value, Firm-Value Risk Model.

JEL Classification. C61

1. INTRODUCTION

Actuarial "optimal dividends models" started with de Finetti (1957) and work on them continues into the present. (For an overview, see Gerber and Shiu 2004). This line of research starts with an inquiry into the optimal strategy for distributing dividends to shareholders based on a stochastic differential equation model of the firm's capital. In this paradigm, the value of the firm is defined as the expected net present value of the dividends it passes on to shareholders until such time as it goes bankrupt. Beyond dividends, other aspects of its capital evolution can be subject to control. Probably the earliest application of this model to risk management is Bather (1969) who posed and solved the problem of optimal proportional reinsurance for Brownian motion capital. Adding risk controls to the model was an active area of research in the 1990s.

In contrast, classical finance theory says that idiosyncratic (i.e., firm-specific) risk is diversifiable and therefore should not matter to the shareholder. Also, the Modigliani-Miller (1958, 1961) theorems state that with frictionless capital markets and a fixed menu of business opportunities, a firm's dividend policy, capital structure, and risk management do not affect the present value of free cash flows, the fundamental determinant of firm value. This is not observed in reality, however, and that fact spawned an extensive literature considering various market "frictions" to explain why capital structure and risk management do, indeed, matter. (See Doherty 2000.)

Froot (2007) summarizes the wider financial research into capital market frictions and the relevance of risk management to a *financial* firm. He presents a three-step model specific to insurance firms which incorporates several key elements: " The first feature is that customers – especially retail policyholders – face contractual performance risks…. Customers are thought to be more risk averse to these product performance issues than are bondholders." Other features include the existence of costs of raising and holding capital and asymmetry of the risk distribution. The objective of his model is to choose initial capital and operating strategies to maximize the ex-ante expectation of market value represented by the payoff from an exogenously given investment opportunity at the 3rd step.

In Major (2008) a connection between Froot's model and optimal dividends models is established, with other of Froot's insights incorporated into a synthesis we term the *Firm-Value Risk Model* (FVRM).[1] In particular, aspects of the evolution of capital are made dependent on the current level of capital. This quickly leads to optimal dividends-type problems that have not been – and perhaps may never be – solved analytically. They can, however, be solved numerically. In doing so, not only can optimal risk management strategies be determined, but their (market) value to the firm can be calculated as well.

In Section 2, I introduce the FVRM and discuss solution methods. Section 3 examines case studies that reveal several "stylized facts" about risk and value in (re)insurance firms. Specifically, the following will be seen:

1. a concave relationship between firm value and financial slack (Froot and Stein 1998)
2. a convex relationship between firm value and risk (Babbell and Merrill 2005)
3. the partial substitution of external capital for reinsurance as the cost of external capital varies (Garven and Loubergé 1996, Garven and Lamm-Tennant 2003, Kavanagh 2006) and
4. the irrelevance of risk management when external capital is freely available (Modigliani & Miller 1958 and 1961, Sethi & Taksar 2002) even in the presence of customer risk aversion (Froot 2007).

I provide conclusions in Section 4.

[1] These had previously been named *FLAVORED* models.

2. FIRM-VALUE RISK MODELS

2.1 Defining an FVRM

An FVRM consists of two parts:

(1) **The Evolution of Wealth Equation** is a stochastic difference or differential equation for book value (surplus, wealth), W, which incorporates capital flows C, D, as well as other management controls, U.

(2) **The Firm-Value Equation** is an expected present value (market value, shareholder value) criterion, M, to be maximized by finding the optimal capital flow and control strategies.

The results from solving such an optimization problem include (a) the relation between surplus and market value, or M-curve, $M(W)$, (b) the optimal capital flow strategy $C(W)$-$D(W)$, and (c) the optimal control strategy $U(W)$. In most theoretical development of optimal dividends models (the underlying technology of FVR modeling), the evolution of wealth equation has been a continuous-time stochastic differential equation. In practice, we have used a finite time step difference equation (Δt = 0.01, 0.25, or 1 year). We may consider a reasonably generic evolution of wealth equation as:

$$W_{t+\Delta t} = W_t + (1-\tau)\cdot\left\{\mu(W_t,u_t) + \iota\cdot(W_t - W_0) - \frac{\gamma}{(1-\tau)}\cdot W_t\right\}\cdot\Delta t + \Delta X(u_t) + \Delta C(u_t) - \Delta D(u_t) \tag{2.1}$$

where W_t is surplus at time t, τ is the tax rate, μ is a profitability offset curve ("P-curve," more about this later), u_t identifies a management control alternative, ι is the pretax investment income rate earned by marginal surplus, W_0 is starting surplus, γ is an assumed growth rate in the scale of operations (exposures and all other proportional financial stock variables), $\Delta X(u_t)$ represents a random shock to surplus conditional on the control value u_t, C represents nondecreasing cumulative external capital inflows, and D represents nondecreasing cumulative dividends (or return of capital) back to shareholders. $W \leq 0$ is an absorbing barrier with $\Delta W = \Delta C = \Delta D = 0$ at all subsequent times.

The P-curve μ is used to represent the phenomenon of customer risk aversion as well as other effects that financial distress may have on profitability. A typical DFA exercise creates simulations starting at $W = W_0$, and the probability distribution of ΔX is based on profitability assumptions for the current financial strength of the firm. But what if the firm were starting with a significantly higher or lower level of surplus? Expected profitability would likely be enhanced or suffer accordingly. Assumptions about this effect are captured in the μ term.

This term acts as an offset from the current situation, so $\mu\,(W_0,u_t) = 0$ for all u_t. The mean of ΔX is therefore modeled as independent of W. Making other features (e.g., moments) of the distribution of ΔX independent of W is not a theoretical requirement for an FVRM, but is convenient and seems reasonable from a practical standpoint.

Considered separately from μ is the impact of the investment income rate ι earned on the most liquid part of capital and the impact of the growth rate γ on the need for capital to support the increasing scale of operations. The γ parameter appears in the value equation (discussed next) to account for future growth in the firm and therefore future growth in dividends, but also in the evolution equation, reflecting the need to add to surplus from retained earnings to support the growing business. The term in braces can be thought of as growth-adjusted expected profitability. In effect, W and M are denominated in a constant-interest bank account numéraire.

The firm-value equation is the solution to the following optimization problem (Bellman Equation):

$$M(w) = \max_{U,C,D}\left\{ \Delta D - (1+\kappa)\cdot \Delta C + \frac{1+\gamma\cdot\Delta t}{1+r\cdot\Delta t}\cdot E\big[M(W_{t+\Delta t})\,|\,W_t = w, U, C, D\big] \right\} \tag{2.2}$$

where κ is a cost load for external capital and r is the hurdle rate used to discount capital flows for equity valuation. In this equation, U represents a *function* of W, and the optimal U is chosen from within a space of such functions. M is enforced to be zero for arguments less than or equal to zero.

The Bellman equation defines the (shareholder, equity market) value of the firm M, when it holds surplus in the amount of w, as the discounted expected value of M itself in the next time step plus or minus immediate capital flows. As explained in Sethi et al. (1991) and Sethi (1996), this is equivalent to the present value of the stream of future capital flows until ruin. Expectations can be taken with respect to a "real-world" probability measure for the evolution of W, with the discount rate r being the appropriate market rate for the firm's equity flows; however, the problematic dependency of r on W makes risk-neutral valuation a preferred approach.

The problem facing the model-builder is to specify the inputs above – the various parameter values (ι, γ, κ, r, τ) as well as the P-curve μ and the conditional distribution of ΔX associated with each risk management state.

2.2 Solving an FVRM

There is a large body of literature addressing the analytical solution of various special cases of the continuous-time FVRM. Jeanblanc-Picqué & Shiryaev (1995) are generally credited with the first rigorous solution of the optimal dividends problem under a Brownian motion risk process. Independently, Asmussen & Taksar (1997) solved the same problem. Radner & Shepp (1996) went a step further and analyzed optimal risk control by allowing the firm to switch between a finite set of mean-variance pairs for the risk process. See Gerber & Shiu (2004) for an historical overview.

Analytical solutions in the case of Brownian risk are usually approached by differentiating the Bellman equation to become the Hamilton-Jacobi-Bellman equation. As is typical with partial differential equations, the solution method involves careful scrutiny of the boundary behavior of M and optimal U, divining their essential properties, correctly guessing their analytical forms, and then teasing out specific values for all the parameters and coefficients. Yong and Zhou (1999) consolidate many of the late-century results in the theory of the Brownian stochastic optimization, in particular where so-called "singular controls" (like dividends here) and nonsmooth "viscosity solutions" are concerned.

When jump risk is included, the principle behind analytical solutions is the same as for Brownian X. Rather than a partial differential equation (PDE), however, now one deals with a partial integro-differential equation (PIDE). Stochastic optimization in the case of jump diffusions is presented in Øksendal and Sulem (2005), where the most general form of the problem (i.e., with continuous and singular controls) is referred to as "Hamilton-Jacobi-Bellman with Quasi-Variational Inequalities."

The prospects for analytical solutions of realistic and detailed FVRM – especially if one wants to incorporate non-analytical distributions generated, say, by a commerical catastrophe model – are not encouraging. Numerical solutions, however, if time-consuming, are straightforward.

Note the definition for M is recursive.[2] Consider the right hand side of the firm value equation to be an operator Λ, taking as argument a proposed M-curve ϕ and returning as its value a new, different, M-curve $\Lambda(\phi)$ defined by:

[2] In Froot (2008) there is a distinction between the M-curve and its ex-ante expectation, the "EM curve." Here, the M-curve is identical to its discounted expectation.

$$\left[\Lambda(\phi)\right](w) = \max_{U,C,D}\left\{\Delta D - (1+\kappa)\cdot\Delta C + \frac{1+\gamma\cdot\Delta t}{1+r\cdot\Delta t}\cdot E\left[\phi(W_{t+\Delta t})\,|\,W_t = w, U\right]\right\} \tag{2.3}$$

From this perspective, it can be seen that the solution for the optimal *M* is a fixed point of this operator, i.e., a solution satisfies $\Lambda(M) = M$. Under mild regularity conditions and with reasonable starting guess ϕ, the sequence

$$\phi,\ \Lambda(\phi),\ \Lambda(\Lambda(\phi)),\ldots,\ \Lambda^{N}(\phi),\ldots$$

converges to the solution *M* (Kushner and Dupuis 2001). Therefore, merely iterating the operator until convergence appears to have taken place will yield the solution. Technically, this is known as the Value Improvement Algorithm and is credited to Howard (1960).

This algorithm requires, at each step of the sequence, finding the optimal *U,C,D* to apply to the calculation of the expectations. Determining them takes time, compared to just applying the operator Γ_{UCD}

$$\left[\Gamma_{UCD}(\phi)\right](w) = \Delta D - (1+\kappa)\cdot\Delta C + \frac{1+\gamma\cdot\Delta t}{1+r\cdot\Delta t}\cdot E\left[\phi(W_{t+\Delta t})\,|\,W_t = w, U\right] \tag{2.4}$$

for a *fixed, known* strategy *U,C,D*. It turns out that if ϕ is a starting *M*-curve, *M* is the solution and *UCD** is the optimal control, then (1) the optimal *UCD* used in evaluating $\Lambda(M)$ is *UCD** and (2) $\phi, \Gamma_{\mathrm{UCD*}}(\phi), \Gamma_{\mathrm{UCD*}}(\Gamma_{\mathrm{UCD*}}(\phi)),\ldots, \Gamma_{\mathrm{UCD*}}^{N}(\phi),\ldots$ converges to *M*. This suggests the following algorithm:

1. Start with a guess *M*-curve ϕ and a guess control UCD(0).
2. For a given UCD(n), iterate $\phi \leftarrow \Gamma_{\mathrm{UCD(n)}}(\phi)$ until convergence is met.
3. Solve for new UCD(n+1) in $\phi \leftarrow \Lambda(\phi)$
4. Repeat from step #2 until convergence in *UCD* is met.

This is known as Howard's Policy Improvement Algorithm. In step 2, the value function corresponding to a particular strategy[3] is found by iterating the faster Γ_{UCD}. Then in step 3, only one application of the optimization operator Λ is used.

In practice, there is no need to run step 2 out to full convergence, because, after all, it is probably not converging to the right answer anyway (unless *UCD*(n) happens to equal *UCD**). Our implementation uses 20 cycles in step 2. Typically, several hundred to several thousand passes through steps 2 and 3 were required for convergence in the case studies dealt with in this paper.

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[3] "Policy" is the usual terminology, but potentially confusing in an insurance context.

Functions M, U, C, and D are rendered on a discrete grid. That is, each function, say M, is represented by an array of values M_k. Each grid location k is associated with a value $W = k\Delta W$. Typically, the grid has on the order of 100 elements. Optimization is carried out on discrete U,C,D values as well.

Convergence is judged by the maximum change in M, U, C, or D over a period of several cycles. Specifically, steps 2 and 3 are run in batches of 100 to 500 cycles with M, U, C, and D arrays written to file (with six decimal places) at the end of each batch. If the file values are identical between batches, convergence is deemed to have occurred. Typical run times for the examples in this paper were on the order of tens of minutes on a 1GHz Pentium processor.

3. CASE STUDIES

3.1 Introduction to the case studies

In this section, I will pose and solve a sequence of increasingly complex examples. In the solutions, we will see certain "stylized behaviors," some of which are to be expected based on the financial economics literature.

In section 3.2, I start with a simple compound Poisson-exponential risk and no external capital (which is analytically tractable) and compare the numerical and analytical solutions. This illustrates the concave relationship between firm value and financial slack suggested by Froot and Stein (1998).

In section 3.3, I introduce a "ratings cliff" (step function) to the underlying profitability μ and add the availability of a premium-loaded excess-of-loss reinsurance layer. A portion of the M-curve is seen to be convex. The reinsurance is seen to add market value even when not used; that is, the *option* to buy reinsurance has value.

In section 3.4, an independent, non-hedgeable Brownian risk is added. By varying the volatility of this risk, we see the convex relationship between firm value and risk suggested by Babbell and Merrill (2005). The adverse impact of this risk on the utilization and value of the XOL reinsurance is also seen.

In section 3.5, the model of section 3.3 is augmented with the possibility of raising costly external capital. By varying the cost parameter, we see a complex pattern of substitution between external capital and utilization of the reinsurance. As suggested by Garven and Loubergé (1996), Garven and Lamm-Tennant (2003), and Kavanagh (2006), less expensive capital is associated with more capital-raising and less reinsurance buying. However, the details of this substitution are quite intriguing. In the limit as external capital becomes freely available, we see convergence to a "Modigliani & Miller world" where risk management has no function and the *M*-curve is a straight line.

3.2 The basic compound Poisson-exponential model

Recall the generic evolution of wealth equation (2.1). We will assume no taxes, no investment income on marginal surplus, and (for now) no external capital infusions. We will approximate continuous time by taking a time step of $\Delta t = 0.01$ year. Risk will be composed of catastrophes occurring at a frequency of λ per year (Poisson distributed) with magnitude of the loss being distributed as (independent) exponentials with mean ν. Other income μ is constant. This version of the evolution equation may be rendered as

$$\begin{aligned} &dW_t = \mu \cdot dt - dX_t - dD_t \\ &dX = J \cdot S \\ &J \sim Poisson(\lambda \cdot dt) \\ &S \sim Exponential(\nu) \end{aligned} \tag{3.1}$$

Gerber and Shiu (2004) present a lucid exposition of the solution logic. They show that if expected income is sufficiently large, the optimal strategy is to distribute immediately all surplus W-β above a "barrier" point β whose value depends on μ, λ, and ν. When W is less than β, no dividends are distributed. The barrier point can be interpreted as the optimal level of capital for the firm: if surplus is above that, excess amounts should be returned to the shareholder immediately;[4] below that, all profits should be retained until the optimal level is reached.

To make the example concrete, and referring to equations 2.1 and 2.2, assume $\tau = \iota = \gamma = 0$ and $r = 1\%$ is the required rate of return. The firm has book value of W_0 = \$9bn and deterministic one-year change in book value of μ = \$1bn if it experiences no natural catastrophes. The cat parameters are $\lambda = 0.5$ and ν = \$1bn. Therefore, expected profits are μ-$\lambda\nu$ = \$500mm.

Figure 1 shows the solution, as calculated by analytical formula (Gerber and Shiu 2006, Albrecher and Hartinger 2007) and by numerical solution.

The analytical solution is the solid line at top. This features a prominent concavity as

[4] Such a strategy might be implemented through stock repurchase.

expected by Froot and Stein (1998) despite the fact that the expected profitability and conditional distributions for the change in surplus are identical at every positive W value. What drives the concavity is (1) the requirement that $W \leq 0$ implies bankruptcy and the termination of all cash flows to shareholders and (2) the fact that there is no possibility of recapitalization. Above the dividend barrier (β = \$10.9bn in this case) the optimal strategy is to dividend back to the barrier, making the M-curve linear ($dM/dW=1$) above the barrier. The M-value at the barrier, 49, is a significant multiple of surplus. Between zero and β, the Bellman expectation operator in effect interpolates a smooth curve.

Numerical solutions in figure 1, from bottom to top, use grid sizes of 30, 60, 120, and 240, respectively, on a range of W from 0 to \$18bn. Figure 2 expresses the numerical error relative to the analytically calculated M value. Over most of its range, the 240-cell solution has less than 1% error. All other numerical solutions in this paper use the 240-cell grid.

3.3 The ratings cliff model

A key innovation in Froot's (2007) model is the representation of customer risk aversion by having the expected profit depend on current surplus.

Continuing the example, say management estimates that if book value were to go below \$5bn, the firm would experience a ratings downgrade that would cause it to experience annual change in book value of only \$250mm, less any catastrophe losses. Therefore, expected profits are \$500mm above the ratings boundary but -\$250mm (i.e., a loss) below the ratings boundary. Raising external capital, above or below the ratings boundary, is still out of the question.

The firm has an opportunity to cede a portion of its catastrophe losses to a reinsurer by way of an excess of loss ("XOL") contract. If losses in a catastrophe exceed \$3bn, the reinsurer will reimburse a fraction U of loss amounts above that up to a maximum reimbursement of \$1bn times U. The firm may choose U between zero and one. The premium for the cover is \$70mm times U per annum. Note that given the assumptions about the occurrence of catastrophes, the actuarial value of the coverage is only \$16mm times U per annum.

We modify the specification of section 3.2 as follows:

$$dW_t = \{\mu(W_t) - U_t \cdot p\} \cdot dt - dX(U_t) - dD_t \tag{3.2}$$

where

$$\mu(W) = 10^9 \cdot \begin{cases} 1, & W \geq 5 \\ 0.25, & W < 5 \end{cases} \tag{3.3}$$

$$dX(u) = 10^9 \cdot J \cdot (S - u \cdot \max(0, \min(S - 3, 1)))$$

$$J \sim Poisson(0.5 \cdot dt)$$

$$S \sim Exponential(1)$$

$$p = 7 \cdot 10^7$$

where the control variable u represents the fraction of coverage purchased. Note that μ is functionally dependent on W but not u, while the probability law for X is functionally dependent on u but not W.

By solving the firm-value equation numerically, we discover that the resulting optimal dividend strategy consists of two zones where dividends should be paid. The "profit" zone extends above the optimal capital level of W = 15.2 ($bn), where all excess capital should be immediately distributed back to the shareholder. The other zone extends below another critical capital level of W = 2.4, where almost all capital should also be immediately distributed back to the shareholder, leaving the firm with only enough to stay in business until the next catastrophe occurs (which would then wipe out capital and cause the firm to go out of business). In between those thresholds, dividends should not be paid, but rather the firm should retain earnings with the goal of increasing capital to the optimal level. The firm, currently at $9bn, is undercapitalized; profits should be retained.

Figure 3 shows the optimal risk transfer strategy. There is a zone 7.8 < W < 10.2 in which risk transfer should be conducted. Between book value of $7.8mm and $8.8mm, the firm should contract to buy 94% coverage. For higher book values, up to $10.2bn, progressively less cover should be purchased. Outside of this range, no cover should be purchased. We may rationalize this as follows: When the firm has relatively large amounts of capital, it can afford to bear a cat loss and the XOL cover is less valuable to it. On the other hand, for book value below $8bn, the coverage leaves an important gap – losses just short of the $3bn retention, which would not be reimbursed, would push the firm over the ratings cliff. For that reason, the reinsurance is much less valuable to the firm and therefore its purchase is not optimal. Note that in W ranges where dividending takes place, the value of U is technically undefined, because the firm will immediately move to another state W', making reinsurance (or any management control) inapplicable at W.

The firm, being at W_0 = $9bn, should purchase 86% coverage.

Figure 4 shows the baseline M-curve, representing the market value without the availability of reinsurance. At this scale of resolution, there is no visible difference between the with- and without-reinsurance curves. For W greater than about 3, the curve rises steeply, then levels off to a 1-for-1 slope above the high dividend barrier of $W = 15.2$. For W between the "going out of business" barrier (2.4) and the ratings cliff (5), the M-curve is convex, representing increasing marginal value of capital as W approaches (from the left) a ratings upgrade. Around the current \$9bn in capital, every new dollar in capital increases firm value by about \$1.7; for lower levels of capital, the rate of market value change accelerates until it reaches a ratio of over 26:1 at the ratings cliff.

Computing the difference between with- and without-reinsurance M-curves gives the value of the XOL cover, as shown in figure 5. The risk-transfer strategy is overlaid for reference. Availability of risk transfer adds \$75mm to the firm currently ($W = 9$), but would be worth as much as \$122mm if the firm were just below \$8bn book value. Note that the mere *availability* of reinsurance brings value, even when the optimal strategy is not to use it. This reflects the value of the *option* to purchase reinsurance.

3.4 The impact of risk

In this section, risk in the model of section 3.3 is augmented by a zero-mean Brownian motion with volatility σ, making the evolution equation:

$$dW_t = \{\mu(W_t) - U_t \cdot p\} \cdot dt - dX_t(U_t) + \sigma \cdot dB_t - dD_t \tag{3.4}$$

We examine the effect of varying σ on firm value and the optimal capital and reinsurance strategies.

Figure 6 shows M-curves for σ in the range of zero to \$5bn per annum. As might be expected, firm value for the well-capitalized firm is adversely affected by earnings volatility. Firm value at surplus of $W = \$10$bn decreases monotonically as σ increases.

The behavior of firm value at lower levels of surplus, in particular below the ratings cliff, is more complex. Recall that with the assumption of no recapitalization, the only way the $\sigma = 0$ firm can regain its rating is to have one or more "lucky" years with no or low-severity cats, sufficient to enable the \$250mm no-cat earnings to increase surplus above the cliff. Positive surplus change in a given year occurs with about 0.676 probability. With the introduction of independent zero-mean Brownian volatility, the firm has an increased chance both to go bankrupt and to rise above the ratings threshold in any given time period. This serves to "smooth" or "interpolate" the convex portion of the *M*-curve;[5] the higher σ, the more smoothing and the less resulting curvature. At any level of surplus under the ratings cliff, there is some positive σ such that firm value is greater than it is when $\sigma = 0$.

This "value of gambling" below the ratings cliff extends over the entire surplus range for higher values of σ as seen in figure 7. A value of σ around 10 leads to the lowest firm value; franchise value is nearly wiped out. For higher σ, however, firm value *rises*.

Figure 8 shows the optimal capital strategies associated with various values of σ. For all values of σ, it is optimal at $W = \varepsilon > 0$ (in this case, the smallest positive grid position is $\varepsilon = 0.075$) to "hope for the best" rather than go out of business immediately. Also, for all values of σ, there is a band – extending from some optimal capital level to infinity – representing the return of excess capital to shareholders. For some values of σ (below 0.5, and between 10 and 20), there is an additional return-of-capital band, as previously seen for $\sigma = 0$, as well. This band represents a shift in strategy from (1) retaining earnings, in the hopes of regaining a favorable rating, to (2) distributing all earnings immediately to shareholders for as long as possible until the default put can be exercised. This is related to the familiar "asset substitution" problem (Fama and Miller 1972), but it is somewhat surprising to see its emergence here in a bimodal fashion. For σ at or above 50, while there is only one dividend band, it starts at $W = \varepsilon$, so it implements the waiting-for-default strategy.

Figure 9 traces the firm value corresponding to surplus W = \$18mm, well above the ratings cliff and all optimal capital levels. It shows a more-or-less flat portion for $\sigma<1$, the collapse of franchise value for $1< \sigma <10$, and the re-emergence of value from the insolvency put for $\sigma>10$. Qualitatively, this is exactly as described in Babbel and Merrill (2005). In particular, see their figure 3.

Figure 10 shows the effect of σ on the optimal reinsurance usage, and figure 11 its effect on the firm value added by reinsurance. For values of $\sigma > 0.8$, no reinsurance is used. For lower values, usage and value above the cliff decrease monotonically with increasing σ.

We see that in the presence of a (fixed) ratings cliff, noise can sometimes add value to the firm. This is not true when μ (the P-curve, expected change in surplus) is a constant because then the *M*-curve is globally concave. With the cliff, "gambling value" is introduced in the region of financial distress where the convexity of the *M*-curve is exploited. This points up a modeling caution: in reality, ratings (or indeed any proxy for customer risk aversion) would

[5] Recall, absent capital flows, the *M*-curve is equal to the discounted value of its own future expectation.

be sensitive to the level of volatility; the cliff should move as a function of σ. Also, if recapitalization were allowed, that would be a superior way to exploit *M*-curve convexity and escape distress. While reinsurance can add value to the firm, that value is hampered by the presence of non-hedgeable volatility.

3.5 The introduction of external capital

In this section we return to the model of 3.3, that is, with no Brownian interference. Preventing recapitalization is equivalent to assuming an arbitrarily large value for the external cost parameter κ. Because the maximum slope of the *M*-curve is around 26, any value of κ greater than 25 (=26-1) will prevent recapitalization from being optimal for any *W* value. However, as the cost parameter decreases, recapitalization becomes increasingly more applicable. Figure 12 shows solid curves, from top to bottom, representing *M*-curves corresponding to κ values of 0.1, 0.7, 1.5, 3, 6, 9, and 30 (which is equivalent to anything over 25), respectively. As κ approaches zero, the *M*-curve straightens out as shown.

A comprehensive view of the effect of different κ values on optimal strategies is given in figure 13. There, *W* is on the horizontal axis and κ is on the (log-scale) vertical axis. The *W* = 5 "ratings cliff" is identified by a vertical dashed line. Various zones are delineated inside the rectangle, indicating where capital flow strategies apply. Contours, indicating optimal reinsurance purchase regions, are also present.

The upper portion of figure 13 corresponds to high-κ values, where external capital is effectively unavailable. Proceeding from left to right, i.e., from low to high values of *W*, we can retrace some of the conclusions presented in the discussion of section 3.3. For values of *W* less than 2.4, the optimal strategy is to wait at $W = \varepsilon$ for the next cat to put the firm out of business. Between 2.4 and 7.8, the firm remains in business but does not buy reinsurance. Between 7.8 and 10.2, various levels of reinsurance cover are optimal. Between 10.2 and 15.4, no reinsurance is purchased. Above 15.2, surplus is given back to shareholders in order to maintain the optimal capital level at 15.2.

These conclusions are the same for all κ values above 25. As slightly lower values are contemplated, there are several changes. The use of recapitalization emerges for *W* values just below the cliff; this is the region of highest slope of the *M*-curve without recapitalization. The lower and upper dividend barriers shift downwards slightly. The zone for purchasing reinsurance also shifts downward. Note that purchasing reinsurance for *W* slightly below 8 now makes sense because falling just below the cliff (*W*=5) can be rectified by recapitalization.

As κ values decrease down to 8, the zone of recapitalization extends to lower *W*, finally eliminating the going-out-of-business zone, and the reinsurance purchase region shrinks and finally disappears at $\kappa = 5.5$. External capital has become cheap enough that reinsurance protection against falling below the cliff is no longer economical.

As κ values fall to 1.5, this status quo is maintained, with the only change being the upper dividend barrier (optimal capital level) falling rapidly.

At $\kappa = 1.5$, a new reinsurance strategy emerges. Reinsurance purchase is indicated at *W* values just above the ratings cliff. This cover does not protect against falling below the cliff – recapitalization takes care of that – it protects against *bankruptcy*. (Recapitalization is not available after bankruptcy in this model.) This strategy was not optimal at lower κ values, but now the franchise value of the firm has risen enough[6] to make it economical to purchase reinsurance for that purpose.

As κ falls from 1 to 0.1, this use of reinsurance expands and the upper dividend barrier continues to fall. However, below 0.1, capital edges out reinsurance yet again – the optimal recapitalization point increases, above the cliff. Now, rather than purchasing reinsurance to protect against bankruptcy, the firm simply makes sure to hold enough capital to withstand losses of a certain size.

As κ approaches zero, a version of the Modigliani-Miller situation obtains. The optimal capital level is a single point, around 7.5, where earnings that cause *W* to move above or below it are adjusted via capital flows to or from shareholders, making the *M*-curve a straight line. This exactly parallels the findings of Sethi & Taksar (2002) in the case of a Brownian motion risk.

As Froot (2007) pointed out, the presence of customer risk aversion alone is not sufficient to invalidate M&M – costly external capital is still required.

4. CONCLUSION

Without a consensus on a value-based theoretical framework, risk management in insurance has been left positing risk-reward preferences to guide appropriate risk transfer, operational, and capital strategies. Such preferences must be elicited from management, who may have only an intuitive grasp of the collective preferences of the firm – assuming such

[6] See figure 12. Shareholder value at the cliff has increased from \$24bn to well over \$40bn, approximately doubling franchise value.

exist. In practice, this means showing managers the risk-return tradeoffs of various decision menu items and letting them "make the call" (Venter 2001). By infusing optimal dividends model formulations with insights from financial economics, the FVRM provides a market perspective of (re)insurer value, obviating the idea of firms having risk preferences – just a preference for higher shareholder value.

While analytical techniques provide insight into the comparative statics of solutions to a gradually widening set of problems, numerical techniques afford the practitioner the means to address more complex and relatively more realistic instances of the model immediately.

This paper demonstrated how a rich set of behaviors, consistent with expectations from the financial economics literature, emerge from FVRMs that incorporate customer risk aversion, nonlinear risk management, and costly external capital.

Acknowledgment

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FIGURES

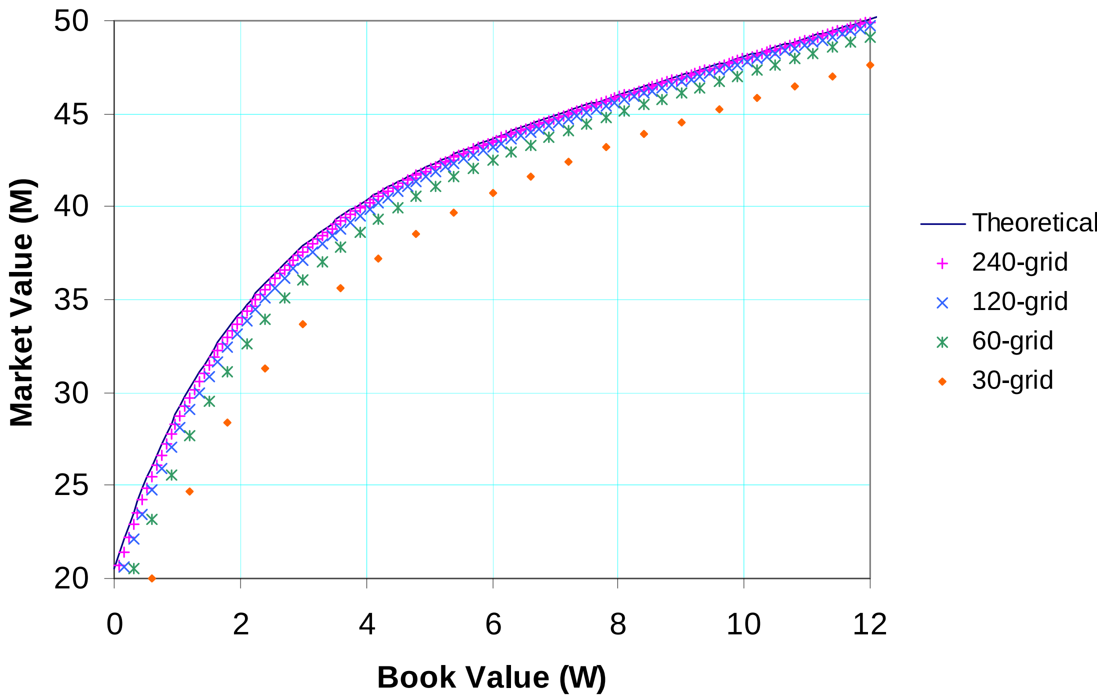


Figure 1: Analytical and numerical solutions of the basic model.

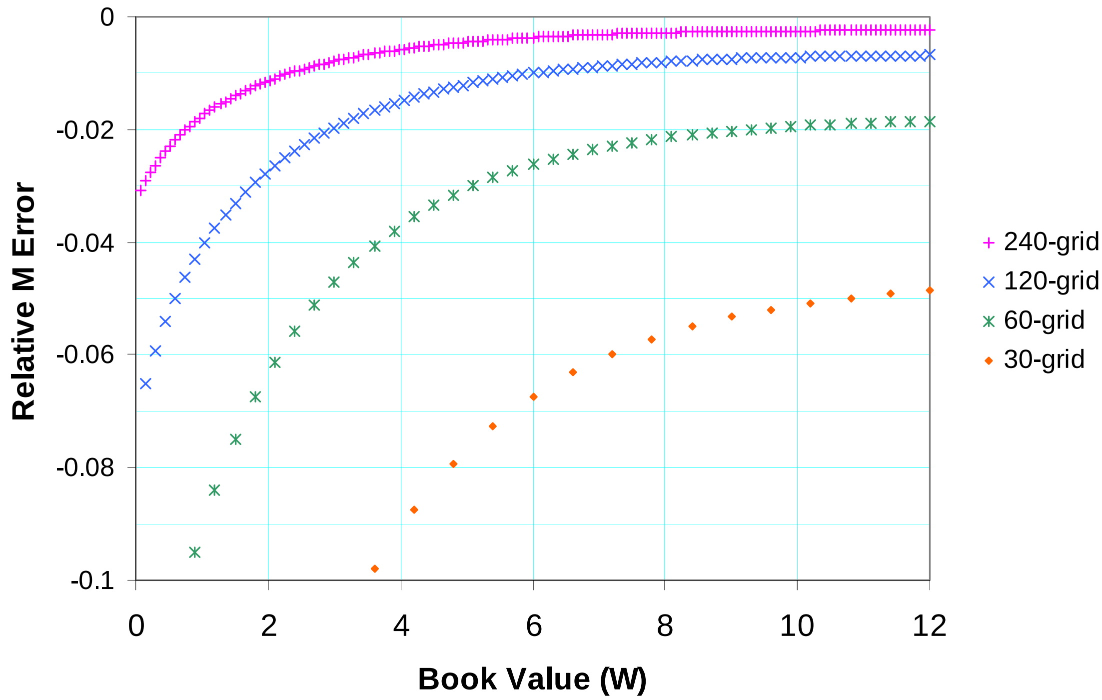


Figure 2: Relative error of numerical solutions.

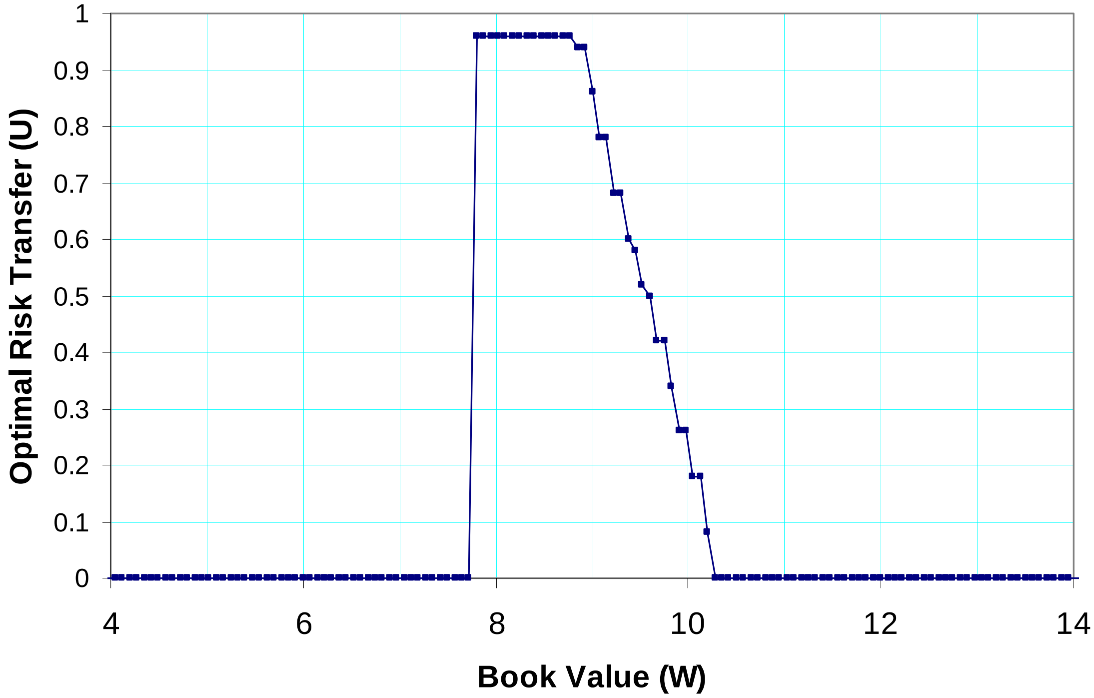


Figure 3: Optimal risk transfer strategy.

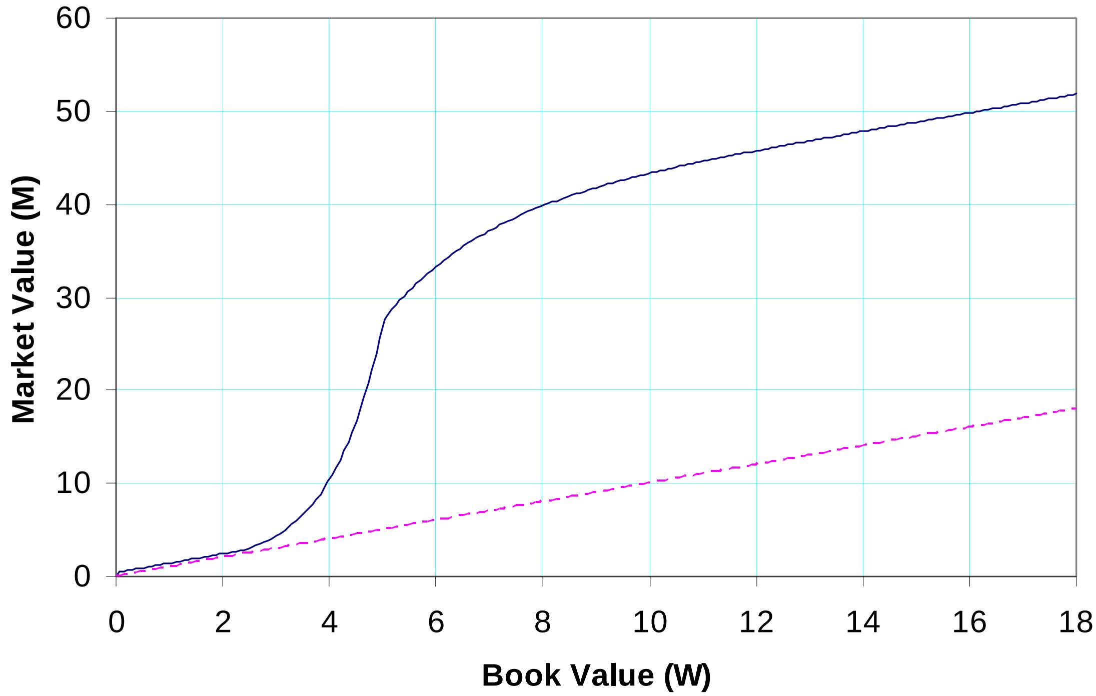


Figure 4: Market value of the firm with ratings cliff.

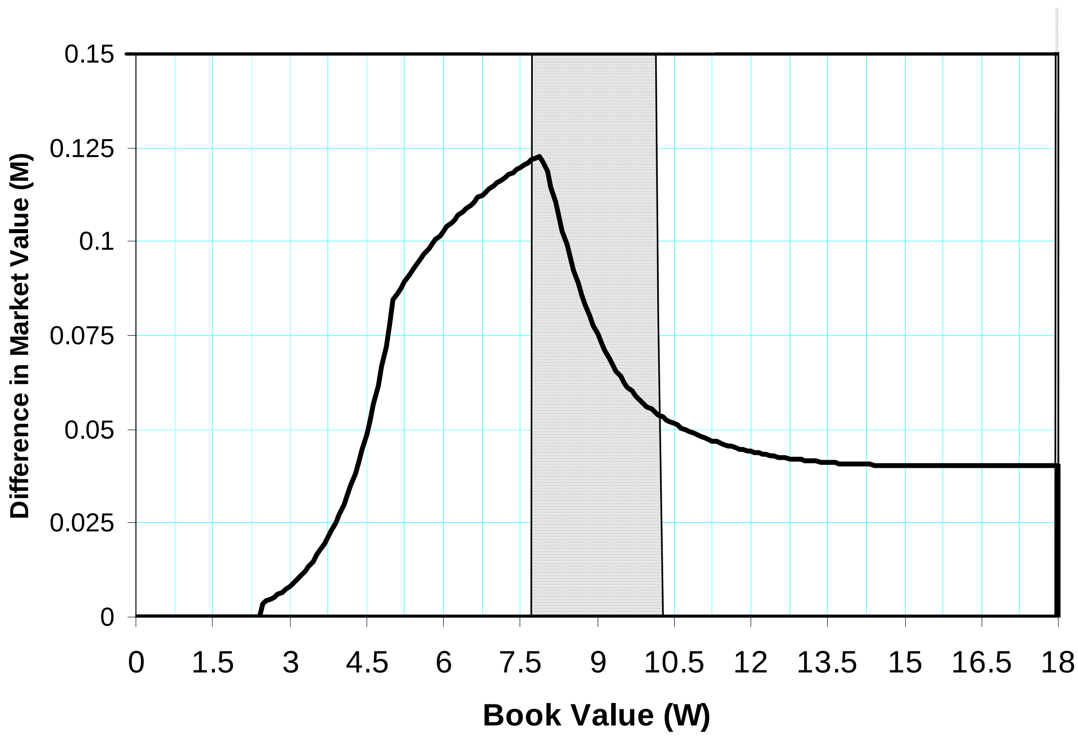


Figure 5: Value of risk transfer.

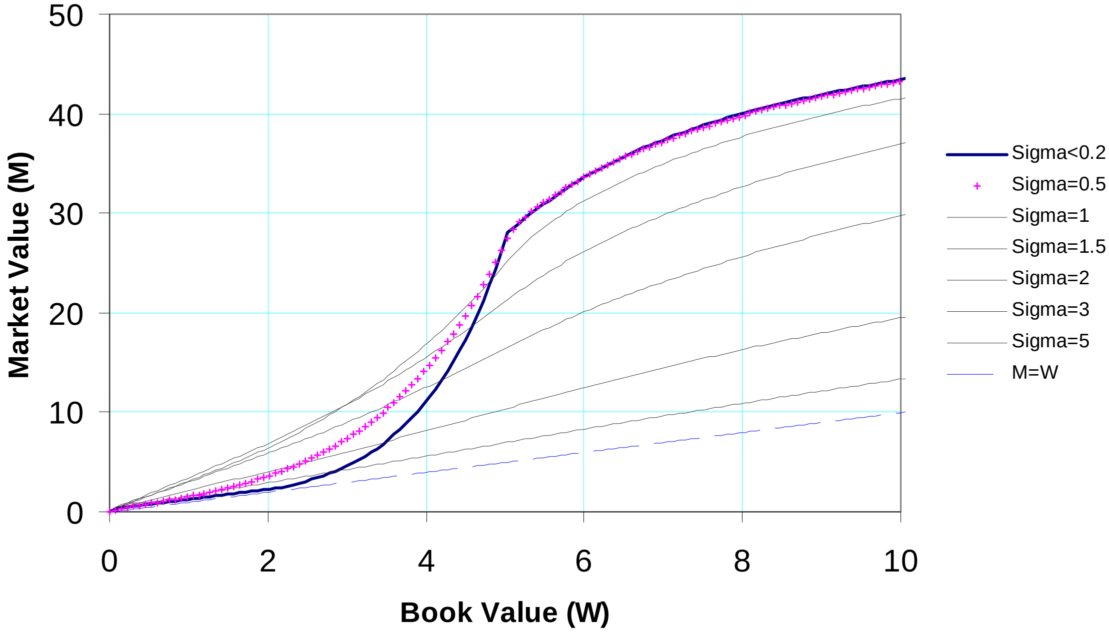


Figure 6: Phase 1 of increasing risk – the collapse of the *M*-curve.

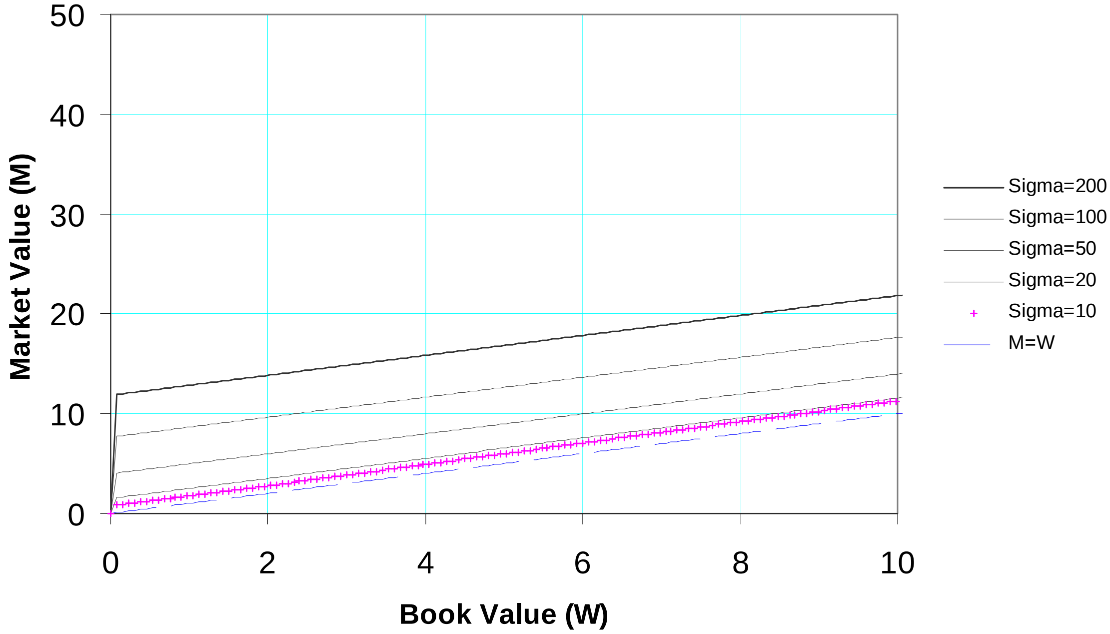


Figure 7: Phase 2 of increasing risk – the rise of the gambling *M*-curve.

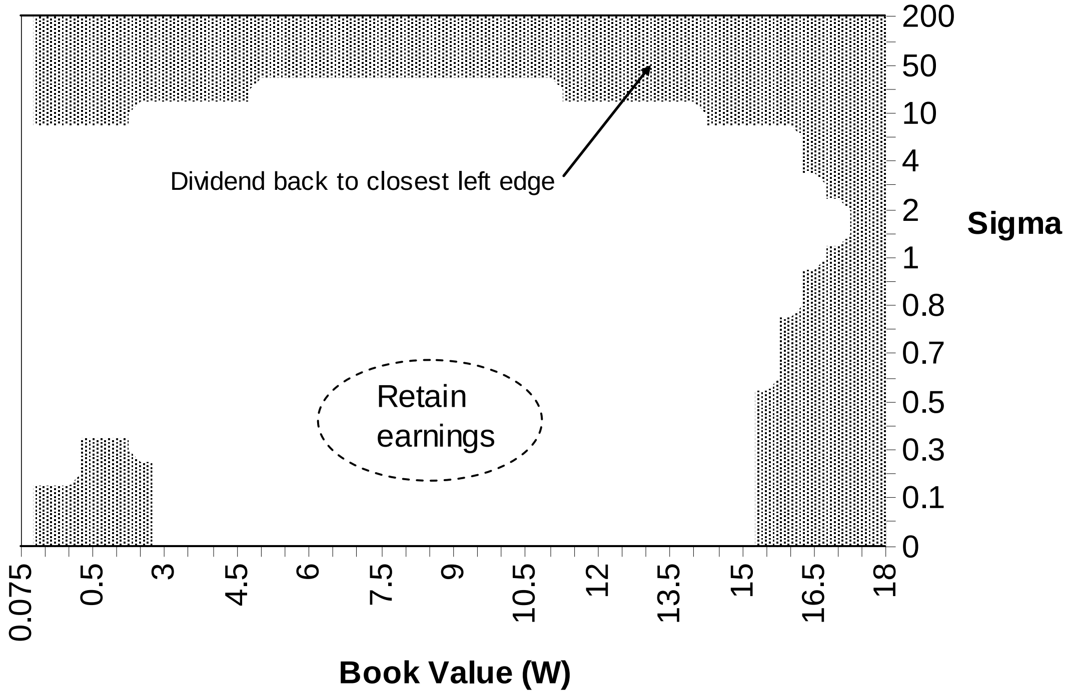


Figure 8: Capital strategy as a function of volatility.

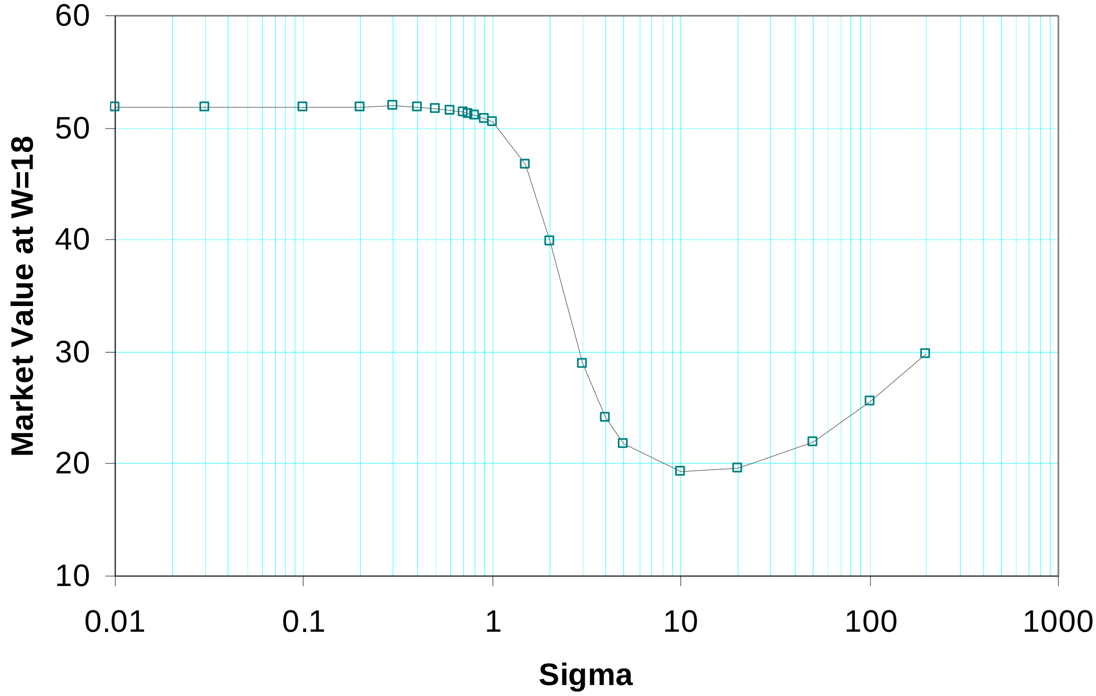


Figure 9: Firm Value as a function of volatility.

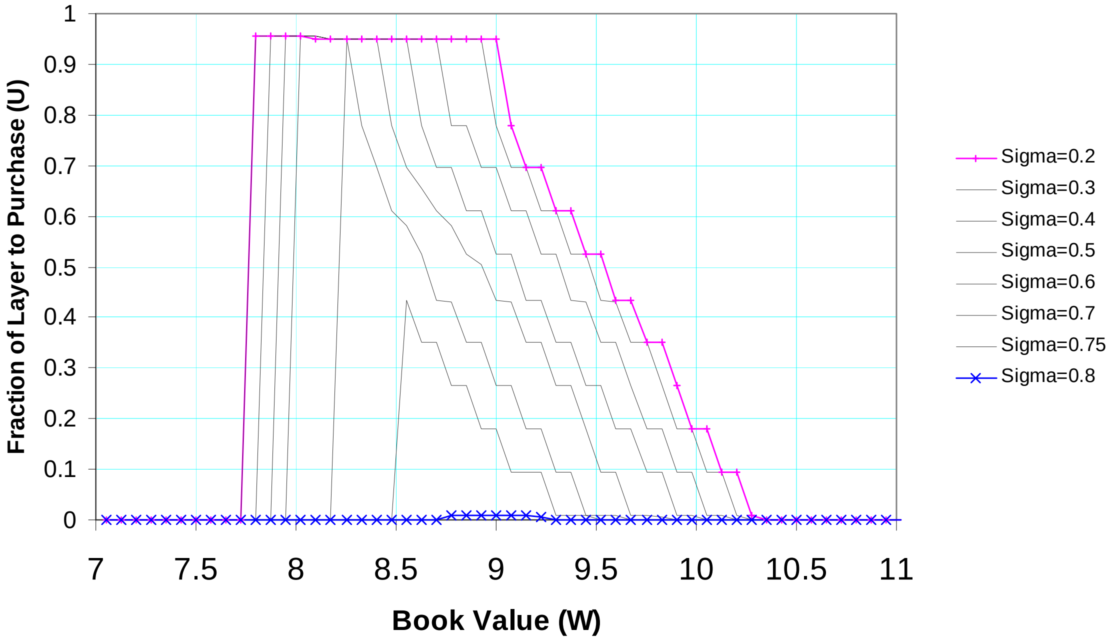


Figure 10: Risk transfer strategy as a function of volatility.

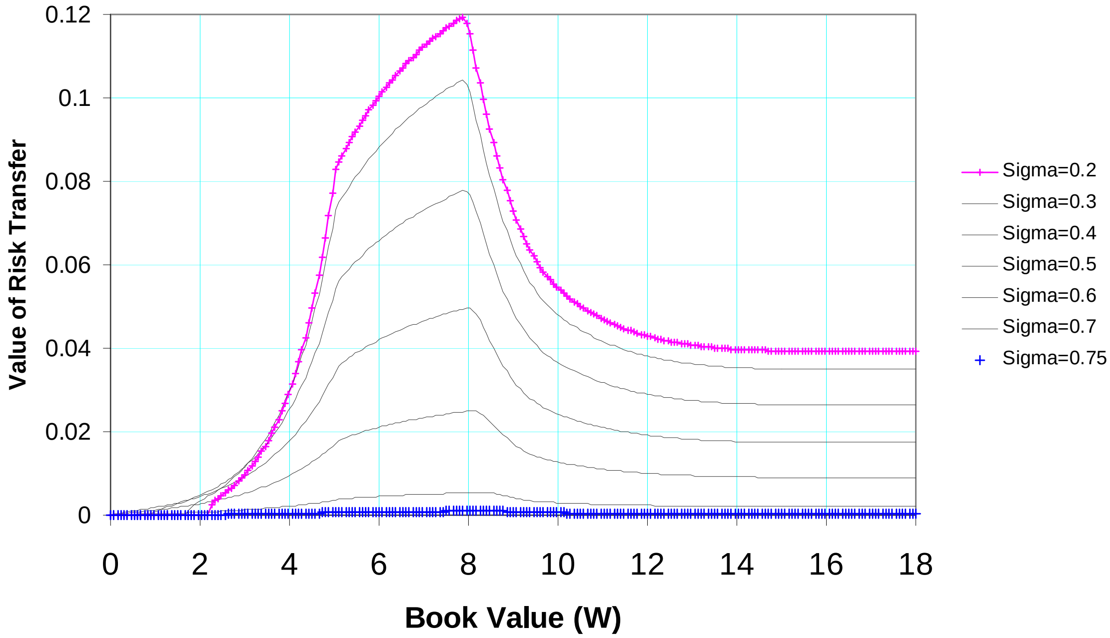


Figure 11: Impact of volatility on value of reinsurance.

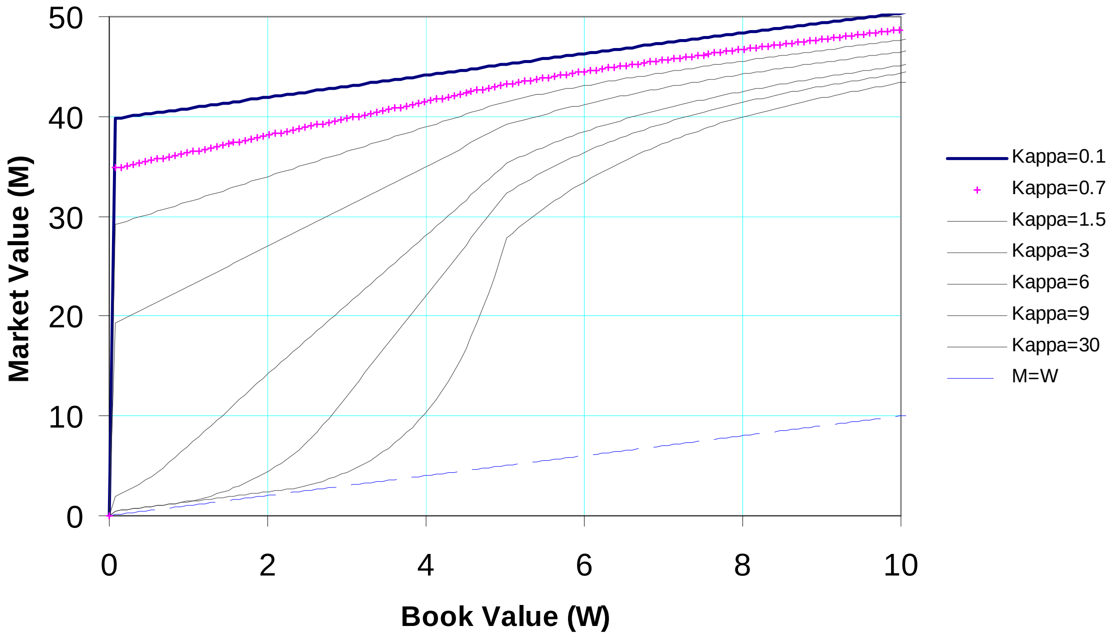
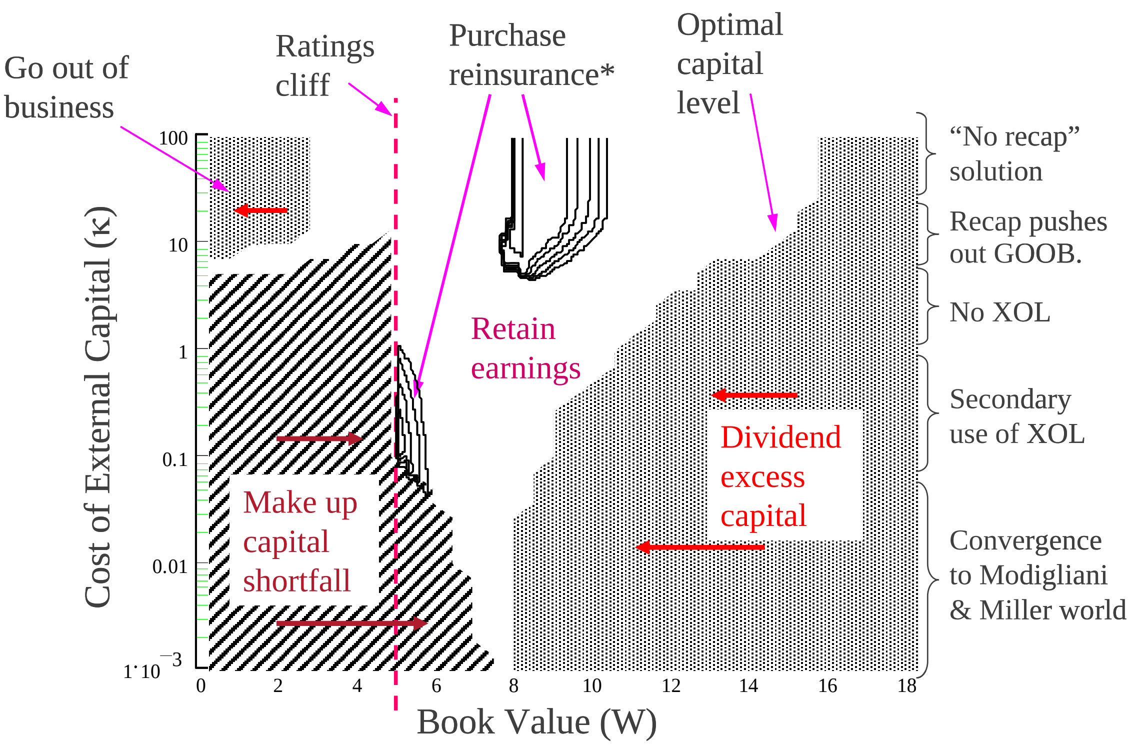


Figure 12: Market value of the firm as cost of external capital varies.



* Reinsurance purchase contours start at 0.1 on outside and increase by 0.2 at each inward step.

Figure 13: Optimal strategies as functions of the cost of external capital.

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